Design Considerations of Junction Transistors at Higher Frequencies

Based Upon an Accurate Equivalent Circuit*

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Summary—**The analytical solution of the diffusion equation has been used to construct an accurate equivalent circuit of the junction transistor. All network components are expressed in terms of the physical parameters of the transistor. This network has been used to calculate Mason's *U*-function, which in turn gives the highest frequency at which a transistor can give a power gain. From the derived formula, it is possible to see how the physical parameters interact to limit the frequency response. The design requirements for a given maximum frequency can be obtained from the formulas; a few examples are discussed. The accurate equivalent circuit will be helpful in designing transistors for more specialized applications.**

* Decimal classification: R282.12. Original manuscript received by the IRE, August 4, 1954. The work for this paper was supported in part by the Bureau of Ships, the Department of the Navy, and the Signal Corps.

This paper was accepted for presentation at the Solid State Devices and Circuits Session of the National Electronics Conference on October 6, 1954.

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In the past, much of the work on junction transistors has been concerned with operation in the audio range. However, during the last year an increasing effort has been made to push the applicability of transistors to higher frequencies.[1-3] This has been done by essentially decreasing the base width, the col-

[1] W. E. Bradley, "Principles of the surface-barrier transistor," Proc. I.R.E., vol. 41, pp. 1702–1706; December, 1953.

[2] J. M. Early, "P-N-I and N-P-I-N junction transistor triodes," *Bell Sys. Tech. Jour.*, vol. 33, pp. 517–533; May, 1954.

[3] C. W. Mueller and J. I. Pankove, "A *p-n-p* triode alloy-junction transistor for radio-frequency amplification," Proc. I.R.E., vol. 42, pp. 386–391; February, 1954.

lector capacitance, and the ohmic base resistance. It is difficult to decrease these quantities simultaneously. In the conventional transistor design, if one of these quantities is made very small, another becomes very large. Compromises become necessary; thus the relative importance of these quantities must be known.

The design problem of a transistor may be divided into three parts. First, the physical processes that determine the properties of the transistor must be understood, so that analytical or numerical results may be obtained for the electrical parameters. Secondly, these results should be re-expressed in as simple an equivalent circuit as possible, so that network calculations can be carried out. Every component of the equivalent network should be expressed as a function of the resistivities, geometry, and other physical parameters. Thirdly, this electrical circuit should be optimized for a specific application by the proper choice of these parameters. In this paper, the limitation of the frequency response by the physical parameters is investigated. The highest frequency at which a transistor can give a power gain is obtained. This calculation can be used to determine, among other things, how large the collector capacity and the base resistance can be made, in order to make full use of a small obtainable base width.

This work can easily be extended to transistors that deviate in some respect from conventional units. It is well known that the emitter capacity in conventional transistors is negligible; thus, it has been omitted in this derivation. However, the large emitter capacity of Early's *n-p-i-n* and *p-n-i-p* transistors[2] can be included, and their highest operating frequency can be calculated. The equivalent circuit given should also be helpful in answering many other questions concerning the operation at high frequencies.

Work on the first part of the design problem overlaps that of Early.[4] The presentation for the grounded base connection at which finally arrived is given in Fig. 1.

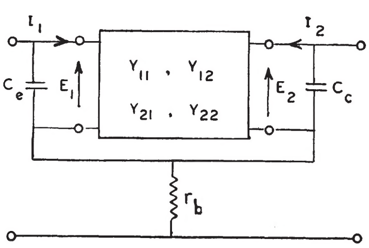

Fig. 1—Schematic representation of junction transistor in grounded base connection.

The following derivation for the equivalent circuit is based on the analytic expressions for the short-circuit admittances of a one-dimensional junction transistor. The point of departure in these studies is the schematic representation in Fig. 1. Units having a low zero frequency α and, therefore, a high dc base current will not be considered here. The voltage generator in series with r_b is neglected and the base resistance is not split into two components. C_c is the "collector capacitance," C_e is the "emitter capacitance," and r_b is the "base spreading" resistance.

The short-circuit driving-point and transfer admittances of the box are defined by the relations[5]

$$Y_{11} = gs_n + \frac{as_p}{\tanh s_p}, \qquad Y_{12} = -\frac{bs_p}{\sinh s_p},$$
$$Y_{21} = -\frac{as_p}{\sinh s_p}, \qquad Y_{22} = \frac{bs_p}{\tanh s_p}, \tag{1}$$

where

$$a = \frac{q}{kT} I_{pe}, \tag{2}$$

$$b = I_{pe}\frac{1}{W_o}\left|\frac{\partial W_o}{\partial V_c}\right|, \tag{3}$$

$$g = a\,\frac{n_n}{p_p}\left(\frac{D_n}{D_p}\times\frac{\tau_p}{\tau_n}\right)^{1/2}, \tag{4}$$

and

$$s_p = \frac{W_o}{L_p}(1 + j\omega\tau_p)^{1/2}; \quad s_n = \frac{W_o}{L_p}(1 + j\omega\tau_n)^{1/2}. \tag{5}$$

In (1) through (5) the following symbols are used:

q = electronic charge,
k = Boltzmann's constant,
T = absolute temperature in degrees Kelvin,
I_{pe} = dc hole current flowing over emitter junction,
W_o = base width at dc operating bias,
V_c = dc collector bias,
n_n = equilibrium concentration of electrons in base region,
p_p = equilibrium concentration of holes in emitter region,
D_n, D_p = diffusion constants of electrons and holes, respectively,
τ_p = lifetime of a hole in the base region,
τ_n = lifetime of an electron in the emitter region.

Numerical values of the physical constants considered here are:

$$\left.\begin{aligned} q/kT &= 40 \text{ volt}^{-1}, \\ I_{pe} &= 10^{-3} \text{ amp}, \\ n_n/p_p &= 10^{-1} \text{ to } 10^{-3}, \\ W_o &\approx 10^{-3} \text{ cm}, \\ D_n &= 90 \text{ cm}^2/\text{sec},\ D_p = 43 \text{ cm}^2/\text{sec}, \\ \tau_n &= \tau_p = 10^{-4} \text{ sec}, \\ L_p &= (D_p \times \tau_p)^{1/2}, \\ \left|\frac{\partial W_o}{\partial V_c}\right| &\approx 2 \times 10^{-5}, \\ r_b &\approx 100 \text{ ohms}, \\ C_c &\approx 10 \text{ to } 100\ \mu\mu\text{f}. \end{aligned}\right\} \tag{6}$$

Using these values, the constants a, b, and g become:

[4] J. M. Early, "Design theory of junction transistors," *Bell Sys. Tech. Jour.*, vol. 32, pp. 1271–1312; November, 1953.

[5] These equations are essentially those of J. M. Early; see ref. 4.

$$\left.\begin{aligned} a &= 4 \times 10^{-2}, \\ b &= 2 \times 10^{-5}, \\ g &= 4\sqrt{2}\,(10^{-3} \text{ to } 10^{-5}). \end{aligned}\right\} \tag{7}$$

The quantity s_p becomes (in the numerical example $s_p = s_n$, because $\tau_p = \tau_n$):

$$s_p = \frac{W_o}{(D_p\tau_p)^{1/2}}(1 + j\omega\tau_p)^{1/2} \approx \sqrt{2}\,10^{-2}(1 + j\omega\tau_p)^{1/2}, \tag{8}$$

and for sufficiently high frequencies where $\omega\tau_p = 10^{-4}\,\omega \gg 1$,

$$\begin{aligned} s_p &\approx \frac{W_o}{\sqrt{D_p}}(j\omega)^{1/2} \approx \sqrt{2}\,10^{-4}(j\omega)^{1/2} \\ &= (1 + j)10^{-4}\sqrt{\omega}. \end{aligned} \tag{9}$$

In order to arrive at a useful network, representation for the box in Fig. 1, (1) may be written as

$$\begin{aligned} Y_{11} &= y + ny_{11}, & Y_{12} &= y_{12}, \\ Y_{21} &= ny_{21}, & Y_{22} &= y_{22}, \end{aligned} \tag{10}$$

with

$$y_{11} = y_{22} = \frac{bs_p}{\tanh s_p}, \tag{11}$$

$$y_{12} = y_{21} = -\frac{bs_p}{\sinh s_p}, \tag{12}$$

$$y = gs_n, \tag{13}$$

$$n = a/b \approx 2000. \tag{14}$$

A two-terminal-pair network is defined with the properties illustrated in Fig. 2. Like an ideal transformer with a ratio n, the input voltage is changed by this factor.

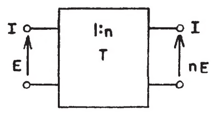

Fig. 2—Two-terminal-pair network representing the active element in the transistor.

Unlike an ideal transformer, however, the current remains unchanged so that the output power is n times the input power. Also, an impedance Z across the output looks like an impedance Z/n at the input, and Z placed across the input looks like nZ when viewed from the output terminals. Circuit-wise this network is as simple to work with as an ideal transformer. From a physical standpoint, it is an active element and represents completely both the active and the non-reciprocal properties of the transistor.[6] The resulting circuit, except for the component shown in Fig. 2, is an ordinary linear, passive, bilateral network.

[6] The same separation into active and passive parts was suggested by J. Zawels at the Transistor Circuit Conference at Philadelphia in February, 1954.

Thus (10), (11), and (12) show that the box in Fig. 1 has the schematic representation given in Fig. 3, in which the linear, passive, bilateral, and symmetrical two-terminal-pair is characterized by the short-circuit admittances of (11) and (12). This network may tentatively be represented physically as a symmetrical lattice with the component admittances

$$y_a = y_{11} - y_{12} = bs_p\frac{\cosh s_p + 1}{\sinh s_p} = bs_p \coth s_p/2, \tag{15}$$

$$y_b = y_{11} + y_{12} = bs_p\frac{\cosh s_p - 1}{\sinh s_p} = bs_p \tanh s_p/2, \tag{16}$$

or with the impedances

$$z_a = \frac{\tanh s_p/2}{bs_p}, \qquad z_b = \frac{\coth s_p/2}{bs_p}. \tag{17}$$

Appropriate network schematics must still be found to represent the box y in Fig. 3 as well as the symmetrical lattice with the impedances of (17).

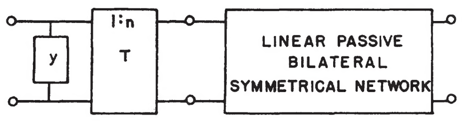

Fig. 3—Schematic network representation of the box of Fig. 1.

For the box y and (8) and (13), it is essential to find an appropriate rational approximation to the irrational expression $(1 + j\omega\tau_n)^{1/2}$. Consider the squared magnitude of the function $(1 + j\bar{\omega})^{1/2}$, which is $(1 + \bar{\omega}^2)^{1/2}$, and plot the logarithm of this squared magnitude versus the logarithm of $\bar{\omega}$ (Fig. 4). This is essentially a straight line rising with a slope of 20 db per decade, for $20 \log (1 + \bar{\omega}^2)^{1/2} \approx 20 \log \bar{\omega}$. A rational approximation to the squared magnitude of $(1 + j\bar{\omega})^{1/2}$ has the form

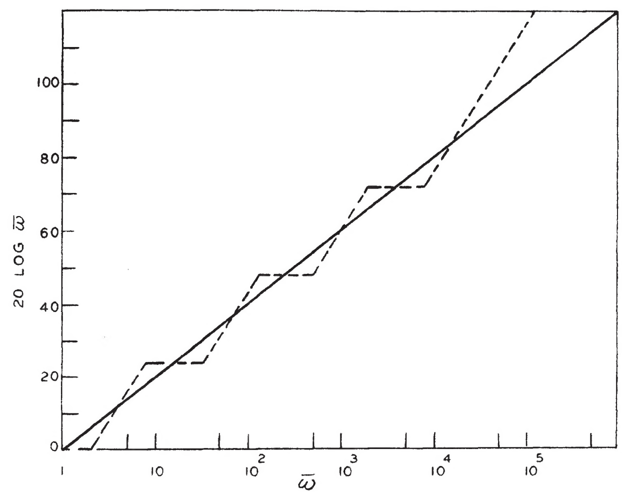

Fig. 4—The squared magnitude of $(1 + j\bar{\omega})^{1/2}$ in decibels versus $\bar{\omega}$.

$$\frac{(1 + \bar{\omega}^2/\bar{\omega}_1^2)(1 + \bar{\omega}^2/\bar{\omega}_3^2)\cdots}{(1 + \bar{\omega}^2/\bar{\omega}_2^2)(1 + \bar{\omega}^2/\bar{\omega}_4^2)\cdots}. \tag{18}$$

Taking 20 log of this expression, the asymptotic slope per factor is 40 db per decade. Hence an approximation may be obtained by alternately rising at the rate of 40 db per decade and zero db per decade, as shown by the dotted line in Fig. 4. Since the actual curve will be smooth and deviate ± 3 db from the dotted line, an approximation that has a maximum deviation of 6 db will yield a Tchebycheff-like approximation with a 3 db ripple.

The break points in this approximation occur at log $\bar{\omega}$ $=0.3, 0.9, 1.5, 2.1, 2.7, 3.3, 3.9$, yielding the critical frequencies: $\bar{\omega}_k=2, 8, 31.6, 126, 500, 2000, 8000$; hence the rational approximation

$$(1+\bar{s})^{1/2} \approx \frac{(\bar{s}+2)(\bar{s}+31.6)(\bar{s}+500)(\bar{s}+8000)}{125(\bar{s}+8)(\bar{s}+126)(\bar{s}+2000)} = y^*. \quad (19)$$

The frequency variable $\bar{s}$ in this expression is to be identified with $j\omega\tau_n$ and, since $\tau_n=10^{-4}$, the highest critical frequency $\omega\tau_n$ in the above rational expression is 8×10^7, corresponding to $\omega=13$ mc. For a lower lifetime material, one would need fewer terms in the expression; for a higher lifetime τ_n, more terms are required.

A network for the admittance function (19) is readily found by expanding $y^*/\bar{s}$ in partial fractions and multiplying by $\bar{s}$ to give

$$y^* = 1 + \frac{2.36\bar{s}}{\bar{s}+8} + \frac{9.85\bar{s}}{\bar{s}+126} + \frac{37.8\bar{s}}{\bar{s}+2000} + \frac{\bar{s}}{125} \cdot \quad (20)$$

The corresponding network is shown in Fig. 5.[7]

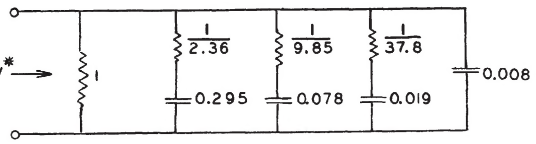

Fig. 5—The normalized equivalent network of the y-box.

Expanding the frequency scale by the factor $1/\tau_n=10^4$, the capacitance values in this circuit become multiplied by $\tau_n=10^{-4}$. The admittance

$$y = \frac{q}{kT} I_{pe} \frac{n_n}{p_p} \left(\frac{D_n}{D_p} \times \frac{\tau_p}{\tau_n} \right)^{1/2} \frac{W_o}{L_p} y^*$$
$$= (10^{-4} \text{ to } 10^{-6}) y^*,$$

and, choosing the intermediate value 10^{-5} for this amplitude factor, the network for y is obtained by multiplying the resistances in Fig. 5 by 10^5 and the capacitances by the additional factor 10^{-5}. The circuit of Fig. 6 is thus obtained for y. The dotted lines indicate points where this network may be cut, and still be an approximate realization for y with the same tolerance when the highest frequency to be considered is less than 50 mc.

Next, the admittance y must be associated with Fig. 3. To do this, y must be divided by n and placed on the

[7] Units of resistance, capacitance, and inductance in the following circuit diagrams are in ohms, farads, and henries, respectively.

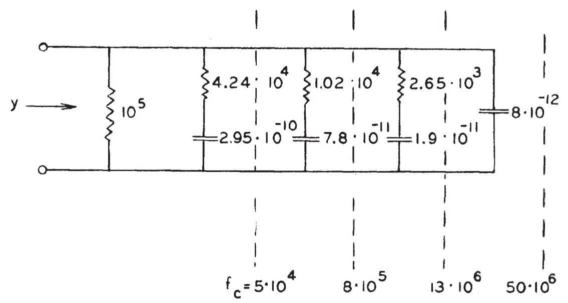

Fig. 6—The equivalent network of the y-box for the given numerical example.

right-hand side of the network T. Since it will be found expedient in the synthesis of the symmetrical network of Fig. 3 to consider the impedances (17) on a level of $2b$ ohms, the admittance y of Fig. 6 should be divided by $2bn=2a=8/100$. In the numerical example it then takes the form shown in Fig. 7.

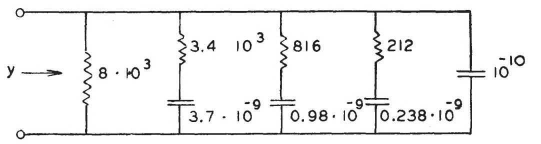

Fig. 7—The circuit of Fig. 6 placed on the right-hand side of the active T-network (on an impedance level of $2b$ ohms).

If the symmetrical network of Fig. 3 is considered in the form of a lattice on a $2b$-ohm impedance level, and the collector capacitance is associated with it, the resultant circuit may be converted into an equivalent T-network as shown in Fig. 8.

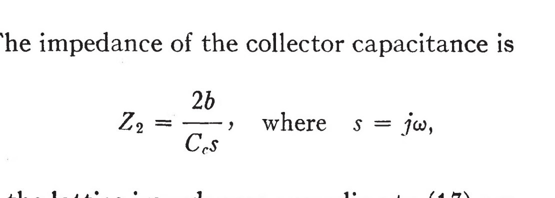

Fig. 8—Conversion of the lattice with associated collector capacity C_c to an equivalent T-network.

The impedance of the collector capacitance is

$$Z_2 = \frac{2b}{C_c s}, \quad \text{where} \quad s = j\omega, \quad (21)$$

and the lattice impedances according to (17) are

$$z_{ao} = 2bz_a = \frac{\tanh s_p/2}{s_p/2}; \quad z_{bo} = 2bz_b = \frac{\coth s_p/2}{s_p/2} \cdot \quad (22)$$

The impedances in the equivalent T-circuit are given by the expressions

$$z_A = \frac{z_{ao}z_{bo} + z_{ao}Z_2}{\frac{1}{2}(z_{bo} + z_{ao}) + Z_2}, \tag{23}$$

$$z_B = \frac{z_{ao}Z_2}{\frac{1}{2}(z_{bo} + z_{ao}) + Z_2}, \tag{24}$$

$$z_C = \frac{\frac{1}{2}(z_{bo} - z_{ao})Z_2}{\frac{1}{2}(z_{bo} + z_{ao}) + Z_2}. \tag{25}$$

Using (22) and (23), and introducing the factor

$$K = \frac{D_pC_c}{bW_o^2} = \frac{D_pC_c}{W_oI_{pe}\left|\frac{\partial W_o}{\partial V_c}\right|} \approx 21.5 \times 10^{11}\, C_c, \tag{26}$$

these impedances may be written in the form

$$z_A = \frac{2K + \frac{\tanh s_p/2}{s_p/2}}{1 + Ks_p \coth s_p}, \tag{27}$$

$$z_B = \frac{\frac{\tanh s_p/2}{s_p/2}}{1 + Ks_p \coth s_p}, \tag{28}$$

$$z_C = \frac{\frac{2}{s_p \sinh s_p}}{1 + Ks_p \coth s_p}. \tag{29}$$

The poles of these functions lie where

$$Ks_p + \tanh s_p = 0. \tag{30}$$

Except for z_C, which also has a pole at $s_p = 0$, the three impedances have identical poles given by

$$\tanh s_p = -Ks_p. \tag{31}$$

Letting

$$s_p = ju, \tag{32}$$

the transcendental equation (31) is transformed into the more easily interpreted form

$$\tan u = -Ku. \tag{33}$$

In considering the roots of this equation, it is significant to note from (26) for K, that, if the collector capacitance C_c is assumed to lie within the range 10^{-11} to 10^{-10} farad, then K has values from 21.5 to 215. Even if the smallest of these values is taken, the roots of (33) are approximately odd multiples of $\pi/2$. For example, if the smallest root (for which the deviation is greatest) assumes the value $(\pi/2)+\delta$, then (33) yields

$$\tan\left(\frac{\pi}{2} + \delta\right) \approx -\frac{1}{\delta} = -K\left(\frac{\pi}{2} + \delta\right);$$

$$\delta \approx \frac{2}{K\pi}. \tag{34}$$

With $K = 21.5$, this gives $\delta = 0.030$ or a 3 per cent deviation.

The pole positions for the impedances (27), (28), and (29) are thus almost independent of the value of C_c within its normal range, and are given by

$$u_\nu = \frac{\nu\pi}{2} \quad \text{for} \quad \nu = 1, 3, 5, \cdots. \tag{35}$$

Let a new frequency variable p be defined as

$$p = \frac{4}{\pi^2}\frac{W_o^2}{D_p}s = 0.97\frac{s}{\omega_\alpha} \approx \frac{s}{\omega_\alpha}, \tag{36}$$

where ω_α is the so-called cut-off frequency. (At this frequency, the ratio of the hole current over the collector junction to the hole current over the emitter junction has dropped to 0.707 of its low-frequency value. As can be shown, $\omega_\alpha = 2.4\ D_p/W_o^2$.)

From (9), (32), and (36),

$$ju = s_p = \frac{W_o}{\sqrt{D_p}}\sqrt{s} = \frac{\pi}{2}\sqrt{p}. \tag{37}$$

All the poles of the impedance functions are seen to lie on the negative real axis of the complex frequency plane at the points

$$p_\nu = -\nu^2 \quad \text{for} \quad \nu = 1, 3, 5, \cdots, \tag{38}$$

or at

$$p_\nu = -1, -9, -25, \cdots.$$

It will be found expedient to express the impedances z_A, z_B, z_C in terms of their partial fraction expansions. To this end, the residues of these impedances in their poles must be computed. In this regard,

$$\frac{d}{du}(1 + jKu \coth ju) \times \frac{du}{dp}\Big]_{u=\nu\pi/2}$$

$$= \frac{d}{du}(1 + Ku \cot u) \times \frac{du}{dp}\Big]_{u=\nu\pi/2}$$

$$\approx (-Ku) \times \left(\frac{-\pi^2}{8u}\right)\Big]_{u=\nu\pi/2} = \frac{K\pi^2}{8}. \tag{39}$$

Thus

$$\text{residues of } z_A = \frac{2K + \frac{\tan \nu\pi/4}{\nu\pi/4}}{K\pi^2/8}$$

$$= \frac{2K + 4(-1)^{(\nu-1)/2}/\nu\pi}{K\pi^2/8} \approx \frac{16}{\pi^2} \tag{40}$$

because $2K \gg |4/\nu\pi|$ for $\nu = 1, 3, 5, \cdots$. Next

$$\text{residues of } z_B = \frac{\frac{\tan \nu\pi/4}{\nu\pi/4}}{K\pi^2/8} = \frac{32(-1)^{(\nu-1)/2}}{K\pi^3\nu}, \tag{41}$$

and

$$\text{residues of } z_C = \frac{\dfrac{-2}{(\nu\pi/2)\sin \nu\pi/2}}{K\pi^2/8} = \frac{\dfrac{-4(-1)^{(\nu-1)/2}}{\nu\pi}}{K\pi^2/8}$$
$$= \frac{32(-1)^{(\nu+1)/2}}{K\pi^3\nu}. \tag{42}$$

In addition for $s_p \to o$ or $p \to o$, the impedance z_C, by (29) and (37), becomes

$$z_C \to \frac{2/s_p^2}{1+K} = \frac{2}{1+K}\frac{4}{\pi^2}\frac{1}{p},$$

from which it is clear that z_C also has a pole at $p=o$ with the residue

$$\frac{8}{\pi^2(1+K)} \approx \frac{8}{K\pi^2}. \tag{43}$$

The impedances (27), (28), and (29) may thus be written

$$z_A = \frac{16}{\pi^2}\left\{\frac{1}{p+1}+\frac{1}{p+9}+\frac{1}{p+25}+\cdots\right\}; \tag{44}$$

$$z_B = \frac{32}{K\pi^3}\left\{\frac{1}{p+1}-\frac{1/3}{p+9}+\frac{1/5}{p+25}-\cdots\right\}; \tag{45}$$

$$z_C = \frac{32}{K\pi^3}\left\{\frac{\pi/4}{p}-\frac{1}{p+1}+\frac{1/3}{p+9}-\frac{1/5}{p+25}+\cdots\right\}. \tag{46}$$

It is useful to note that

$$z_A(o) = 2;\ z_B(o) = \frac{1}{K};\ \left[\left(z_C - \frac{8}{K\pi^2 p}\right)\right]_{p=o} = -\frac{1}{K}. \tag{47}$$

If the infinite expansions (44), (45), and (46) are terminated at some finite point, the approximation will be improved if a constant term appropriate to the requisite behavior in the vicinity of zero frequency is added.

In this connection, according to (36), a frequency range from $\omega=o$ to $m\omega_\alpha$ corresponds to a p range from o to jm. For most purposes, therefore, only a few terms in the above partial fraction expansions will suffice to obtain an equivalent circuit representation with very good accuracy up to several times cut-off frequency.

For example, using three terms,

$$z_A = \frac{16}{\pi^2}\left\{\frac{1}{p+1}+\frac{1}{p+9}+\frac{1}{p+25}+0.083\right\}; \tag{48}$$

$$z_B = \frac{32}{K\pi^3}\left\{\frac{1}{p+1}-\frac{1/3}{p+9}+\frac{1/5}{p+25}-0.002\right\}; \tag{49}$$

$$z_C = \frac{32}{K\pi^3}\left\{\frac{\pi/4}{p}-\frac{1}{p+1}+\frac{1/3}{p+9}-\frac{1/5}{p+25}+0.002\right\}. \tag{50}$$

The equivalent circuit for the transistor (except for the box T and the admittance y of Fig. 3, and the base resistance $2br_b$) has, for three terms, the appearance shown in Fig. 9.

It is interesting to note that the sum of z_B and z_C reduces to the capacitive reactance $8/K\pi^2 p = 2b/C_{cs}$ of the collector junction alone. Comparison with the circuit in Fig. 7 for the admittance y shows that the latter is entirely negligible for most practical purposes. In cases where it is not negligible, it can easily be taken into account.

In order to return to an ordinary impedance level, the values of all resistances must be divided by $2b$ and the values of all capacities multiplied by $2b$. In addition, by returning to the ordinary frequency scale, the capacities must be divided by ω_α. Table I gives the factors by which the resistances and capacitances of Fig. 9 must be multiplied in the completely denormalized circuit.

TABLE I

Branch	z_A	z_B	z_C
Resistances	$\frac{8}{\pi^2}\frac{1}{b}$	$\frac{16}{\pi^3}\frac{1}{bK}$	$\frac{16}{\pi^3}\frac{1}{bK}$
Capacitances	$\frac{\pi^2}{8}\frac{b}{\omega_\alpha}$	$\frac{\pi^3}{16}\frac{Kb}{\omega_\alpha}=0.81C_c$	$\frac{\pi^3}{16}\frac{Kb}{\omega_\alpha}=0.81C_c$

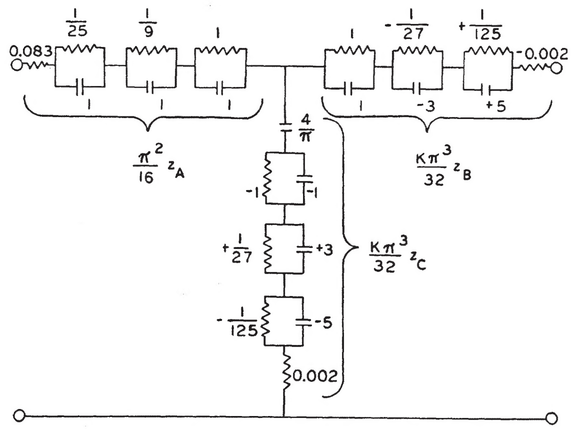

Fig. 9—The explicit form of the T-network of Fig. 8 (three terms in the approximation).

This equivalent circuit will be used to compute the unilateral gain[8] of the transistor as a function of frequency and of its physical parameters. In terms of the over-all (open-circuit driving-point and transfer) impedances Z_{11}, Z_{22}, Z_{12}, and Z_{21}, and their real parts R_{11}, R_{22}, R_{12}, and R_{21}, this quantity is defined by

$$U = \frac{|Z_{12}-Z_{21}|^2}{4(R_{11}R_{22}-R_{12}R_{21})}. \tag{51}$$

Since $Z_{12} = z_C/n + 2br_b$ and $Z_{21} = z_C + 2br_b$,

$$|Z_{12}-Z_{21}|^2 = \left(\frac{n-1}{n}\right)^2 |z_C|^2 \approx |z_C|^2. \tag{52}$$

[8] S. J. Mason, "Power Gain in Feedback Amplifiers," Technical Report No. 257, R.L.E., Mass. Inst. Tech.; August, 1953.

To compute the pertinent real parts, first,

$$z_A = r_A + jx_A, \quad z_B = r_B + jx_B, \quad z_C = r_C + jx_C, \tag{53}$$

then

$$R_{11} = \frac{r_A + r_C}{n} + 2br_b, \qquad R_{12} = \frac{r_C}{n} + 2br_b,$$
$$R_{22} = r_B + r_C + 2br_b, \qquad R_{21} = r_C + 2br_b, \tag{54}$$

hence

$$R_{11}R_{22} - R_{12}R_{21} = \frac{1}{n}(r_Ar_B + r_Ar_C + r_Br_C) + 2br_b\left(\frac{r_A}{n} + r_B\right). \tag{55}$$

Since $r_B+r_C=0$, it follows that $r_Ar_B+r_Ar_C+r_Br_C=r_Br_C=-r_B^2=-r_C^2$. But r_B^2 or r_C^2 is proportional to $1/K^2$, and terms in $1/K^2$ were neglected in determining the poles of the functions z_A, z_B, z_C. Consistency with this approximation requires that the term $(r_Ar_B+r_Ar_C+r_Br_C)$ be dropped therefore

$$U = \frac{|z_C|^2}{8br_b\left(\frac{r_A}{n} + r_B\right)} = \frac{r_C^2 + x_C^2}{8br_b\left(\frac{r_A}{n} + r_B\right)}. \tag{56}$$

With $p=jx$ we readily find from (49), (50), and (51),

$$r_A=\frac{16}{\pi^2}\left[\frac{1}{1+x^2}+\frac{1/9}{1+x^2/81}+\frac{1/25}{1+x^2/625}+0.083\right], \tag{57}$$

$$r_B=-r_C=\frac{32}{K\pi^3}\left[\frac{1}{1+x^2}-\frac{1/27}{1+x^2/81}+\frac{1/125}{1+x^2/625}-0.002\right], \tag{58}$$

$$x_C=\frac{-8}{K\pi^2x}\left[1-\frac{4}{\pi}\left(\frac{x^2}{1+x^2}-\frac{x^2/243}{1+x^2/81}+\frac{x^2/3125}{1+x^2/625}\right)\right]. \tag{59}$$

Introducing the functions

$$X_1(x)=\frac{-\pi^2}{8}Kx_Cx = 1-\frac{4}{\pi}\left[\frac{x^2}{1+x^2}-\frac{x^2/243}{1+x^2/81}+\frac{x^2/3125}{1+x^2/625}\right], \tag{60}$$

$$R_1(x)=Kr_B=-Kr_C = \frac{32}{\pi^3}\left[\frac{1}{1+x^2}-\frac{1/27}{1+x^2/81}+\frac{1/125}{1+x^2/625}-0.002\right], \tag{61}$$

$$R_2(x)=\frac{r_A}{2}=\frac{8}{\pi^2}\left[\frac{1}{1+x^2}+\frac{1/9}{1+x^2/81}+\frac{1/25}{1+x^2/625}+0.083\right], \tag{62}$$

then

$$\left(\text{noting from (37) that } x = \frac{4}{\pi^2}\frac{W_o^2\omega}{D_p}\right)$$

U becomes

$$U = \frac{D_p}{2W_o^2r_bC_c\omega^2}F(x) = \frac{\omega_\alpha}{4.8r_bC_c\omega^2}F(x), \tag{63}$$

where

$$F(x) = \frac{X_1^2 + \frac{\pi^4}{64}x^2R_1^2}{R_1 + \frac{2K}{n}R_2}. \tag{64}$$

This factor equals $(1+2K/n)^{-1}\approx 1$ for $x=O$ (since $X_1(o)=R_1(o)=R_2(o)=1$) and depends upon physical parameters only through the quantity

$$\frac{K}{n} = \frac{Kb}{a} = \frac{D_pC_c}{W_o^2\frac{q}{kT}I_{pe}} = \frac{\omega_\alpha C_c}{2.4\frac{q}{kT}I_{pe}}, \tag{65}$$

since the functions X_1, R_1, R_2 are pure numerics.

Once the factor $F(x)$ is plotted versus x for several values of K/n, the unilateral gain of the transistor as a function of frequency, and for any values of its physical constants, is readily computed.

The functions X_1, R_1, and R_2 have been plotted in Figs. 10 to 12 as a function of $x\approx\omega/\omega_\alpha$ using one to three terms in the expansions (44) to (46). It can be seen that all three approximations agree well up to $x=2.5$ or $\omega\approx 2.5\omega_\alpha$. For most practical purposes the approximation using the first term only will therefore be adequate. The function $F(x)$ depends on the parameter K/n. $F(x)$ is plotted in Figs. 13 to 15 for the values $K/n=1.075\times10^{-2}$, 2.5×10^{-2}, and 10.75×10^{-2} using all three approximations. Again, the first approximation is adequate up to $x=2.5$ or $\omega\approx 2.5\omega_\alpha$. Furthermore, it is interesting to note that $F(x)$ is very close to unity for $K/n=1.075\times10^{-2}$ and 2.5×10^{-2}, up to $w=3\omega_\alpha$. For $K/n=10.75\times10^{-2}$, $F(x)$ is still reasonably constant over the same frequency range, but its value varies from 0.8 to 0.7. In the numerical example given, the value of $K/n=2.5\times10^{-2}$ corresponds to a collector capacity of about 23 $\mu\mu$f. Remembering that the base width in the example was 10^{-3} cm, it becomes obvious that for many practical cases the assumption $F(x)=1$ is a sufficient approximation.

As has been outlined,[8] the U-function yields the highest frequency at which a transistor in any connection, and in any external circuit can give a power gain. The

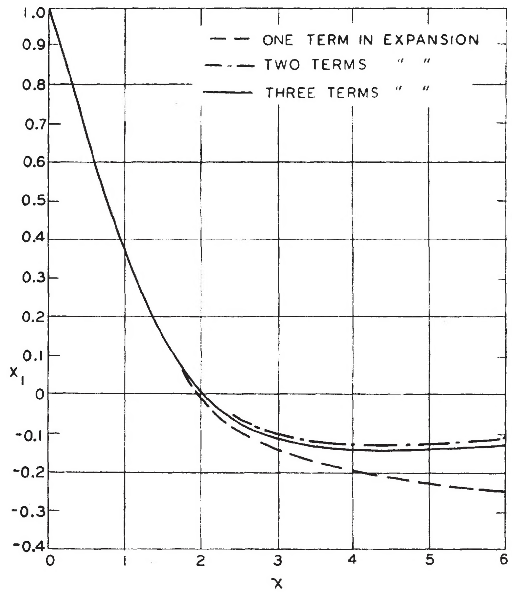

Fig. 10—The function X_1 versus $x \approx \omega/\omega_\alpha$.

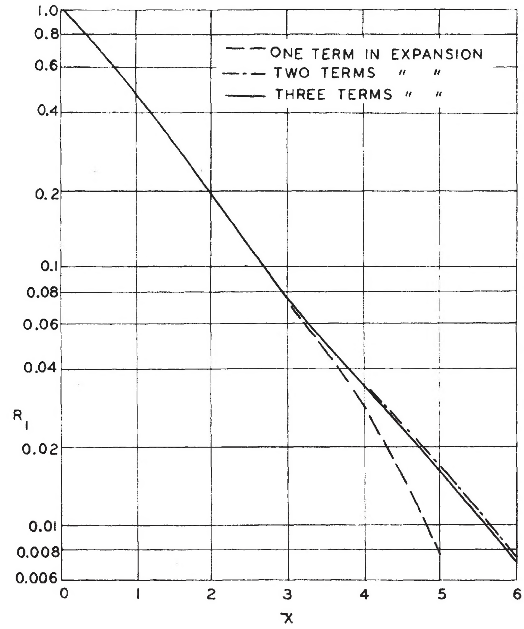

Fig. 11—The function R_1 versus $x \approx \omega/\omega_\alpha$.

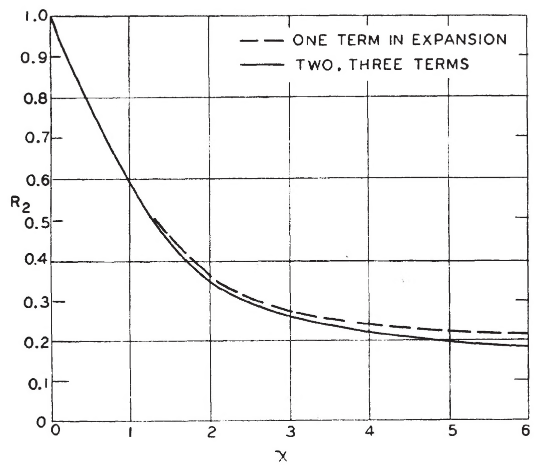

Fig. 12—The function R_2 versus $x \approx \omega/\omega_\alpha$.

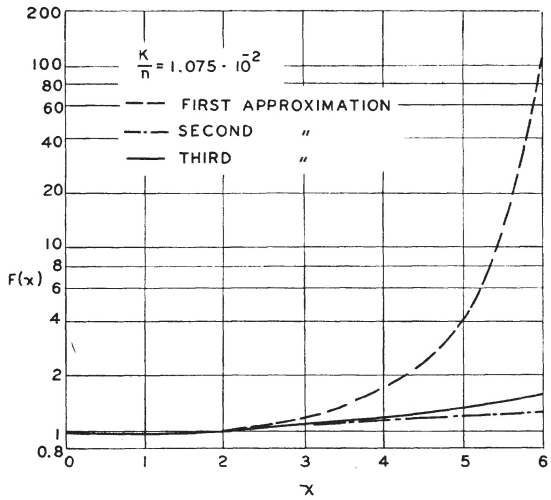

Fig. 13—The function $F(x)$ versus $x \approx \omega/\omega_\alpha$ for $K/n = 1.075 \times 10^{-2}$.

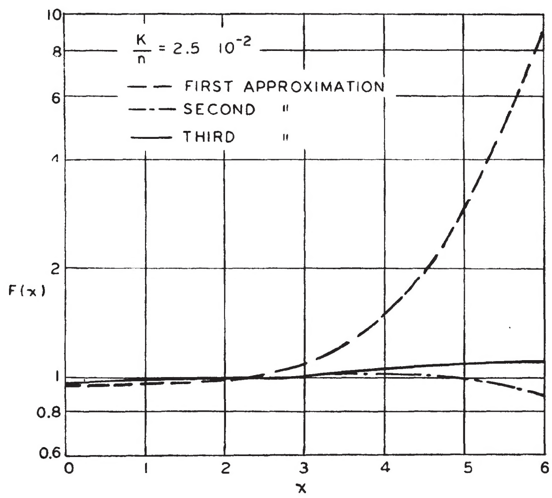

Fig. 14—The function $F(x)$ versus $x \approx \omega/\omega_\alpha$ for $K/n = 2.5 \times 10^{-2}$.

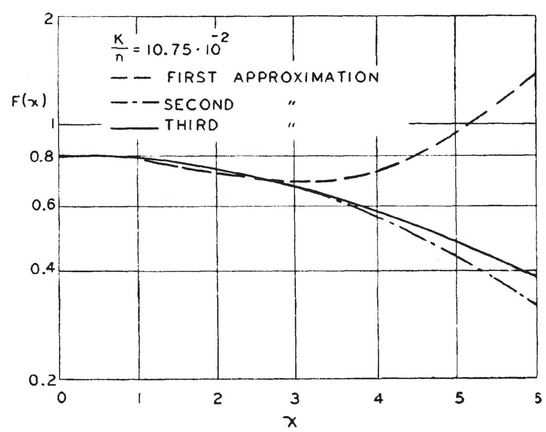

Fig. 15—The function $F(x)$ versus $x \approx \omega/\omega_\alpha$ for $K/n = 10.75 \times 10^{-2}$.

restrictions given there[8] do not apply here. The frequency at which U is equal to unity represents the limiting frequency.

Consider the following example. In a practical circuit, for a transistor to be used up to the cut-off frequency ω_α, the gain U must go through unity at a frequency higher than ω_α. Let this frequency be $\omega = 2\omega_\alpha$, and $F(x)$ assumed to be one. Table II gives values of the product $r_b C_c$ for various frequencies f_α. C_c is determined by choosing a reasonable value of r_b in the conventional transistor designs.

TABLE II

f_α	$r_b C_c$ (ohms f)	Assumed r_b (ohms)	Calculated C_c ($\mu\mu$f)
1 mc	8.3×10^{-9}	100	83
10 mc	8.3×10^{-10}	300	2.8
100 mc	8.3×10^{-11}	1000	0.083
1000 mc	8.3×10^{-12}	3000	0.0028

From Table II it becomes apparent that the conventional transistor will have its practical limit in the range of 100 mc. A fused collector with 0.083 $\mu\mu$f on 1 ohm cm n-type material at 6 v bias would have a diameter of 4.3×10^{-3} cm. The p-n-i-p and n-p-i-n transistor suggested by Early[2] might go to higher frequencies. These transistors were not discussed at the present time because the effect of the emitter capacity (which is not negligible in the above-mentioned transistor) was not included. Table II is correct only if throughout the entire range of values, the simplifying assumptions $F\alpha = 1$ and $K \gg 1$ are justified. It can be shown that $K \gg 1$ means small injection levels for alloy-junction transistors. For extremely small cross sections (which are required for very small capacities), the small injection condition is easily violated.

It is interesting to note that the quantity $|\partial W_o / \partial V_c|$ does not appear in the final expression for U and, therefore, has no influence on the highest operating frequency.

Acknowledgment

The authors wish to thank Miss J. Sullivan for checking the approximations against the correct formulas and for help in preparing the manuscript.